VALUES OF GAMES WITH A CONTINUUM OF PLAYERS*

BY

YAKAR KANNAI

ABSTRACT

A definition of the "Shapley value" of games with a continuum of players and a formula for this value are given for a certain class of games, regarding them as limits of games with a finite number of players.

Introduction. The notion of "the Shapley value" of an n-person game is well-known [3]. If the game is given in the characteristic function form, it is possible to write down a formula for the value. Here we shall treat the value problem for games with a continuum of players.

A definition and a formula were given for certain such games by R. J. Aumann and by L. S. Shapley in an unpublished work, using invertible strongly mixing measure-preserving transformations. In this paper, we shall show that the same formula can be obtained by regarding the continuous game as a "limit", in suitable sense, of a sequence of finite games.

The characteristic function of an n-person game is a real-valued function v, defined on the subsets S of the set $N = \{1, \cdots, n\}$ of the players. The Shapley value $w(i)$ of the player i is given by

$$w(i) = \sum_{S \subseteq N} \frac{(n-s)!(s-1)!}{n!} [v(S) - v(S - \{i\})] \tag{1.1}$$

where s is the cardinality of S.

The characteristic function of a game with a continuum of players is a real valued function v, defined on the measurable subsets of $[0, 1]$. In this case, there is no immediate analogue of (1.1).

Let $f(x_1, \cdots, x_k)$ be a C^1 function of the k real variables $x_1, \cdots, x_k$ in the rectangle $0 \leqq x_1 \leqq a_1, \cdots, 0 \leqq x_k \leqq a_k$. Let $\mu_1, \cdots, \mu_k$ be non-negative measures on $[0, 1]$, absolutely continuous w.r.t. Lebesgue measure, such that $\mu_i([0, 1]) = a_i$. Let the characteristic function v be given by

$$v(T) = f(\mu_1(T), \cdots, \mu_k(T)), T \subseteq [0, 1]. \tag{1.2}$$

Received February 27, 1966.

* The research described in this paper was partially supported by the U.S. Office of Naval Research, Logistics and Mathematical Statistics Branch, under Contract Number N 62558-3586.

This case was treated by Aumann and Shapley. Using invertible strongly mixing measure-preserving transformations, the formula

$$(1.3)\qquad w(T) = \int_0^1 \sum_{i=1}^{k} \mu_i(T) f_i(a_1x, \cdots, a_kx)dx, \left(f_i = \frac{\partial f}{\partial x_i}\right)$$

was obtained for the Shapley value $w(T)$.

We shall arrive at (1.3) in another way.

Let $T \subseteq [0,1]$ and let us be given a dense decreasing sequence of partitions [2, pp. 171, 172] of $[0,1]$ ($[0,1] = \bigcup_{i=1}^{n(m)} A_i^{(m)}$) such that $T = \bigcup_{k=1}^{k(m)} A_{i_j}^{(m)}$. It is well-known ([2, p. 172]) that in this case $\lim_{m\to\infty} \max_{1\leqq i\leqq n(m)} \mu(A_i^{(m)}) = 0$. For every m, consider the $n(m)$-person game whose characteristic function is given by $v_m(\{i_1,\cdots,i_s\}) = v(\bigcup_{j=1}^{s} A_{i_j}^{(m)})$. Define $w_m(A_i^{(m)})$ $(1 \leqq i \leqq n(m))$ to be the Shapley value of the i-th player for this m-th game. According to (1.1),

$$(1.4)\qquad w_m(A_i^{(m)}) = \sum_{S\subseteq N} \frac{(n(m)-s)!(s-1)!}{n(m)!} [v_m(S) - v_m(S - \{i\})]$$

where N is the set $\{1,\cdots,n(m)\}$ and s is the cardinality of S. If $R = \bigcup_{j=1}^{r} A_{i_j}^{(m)}$, define $w_m(R)$ by $w_m(R) = \sum_{j=1}^{r} w_m(A_{i_j}^{(m)})$. If the limit $\lim_{m\to\infty} w_m(T) \underset{\text{def}}{=} w(T)$ exists, we shall define the Shapley value of the game v to be equal to w. This limit does exist in the case treated by the following

THEOREM. *If* $v(T) = f(\mu_1(T),\cdots,\mu_k(T))$, *where* $f \in C^1([0,a_1] \times \cdots \times [0,a_k])$ *and* μ_i *are absolutely continuous non-negative measures with* $\mu_i([0,1]) = a_i$, *then* $\lim_{m\to\infty} w_m(T)$ *exists for all measurable* T *and is given by* (1.3).

The author wishes to thank R. J. Aumann and M. A. Perles for some very helpful conversations concerning this theorem. Also, M. A. Perles has privately communicated to the author various extensions of the results of this paper.

2. **Proof of the Theorem.**

Let us write $n(m) = n$ and omit the superscript m.

$$\begin{aligned} w_m(A_i) &= \sum_{S\subseteq N} \frac{(n-s)!(s-1)!}{n!} [v_m(S) - v_m(S-\{i\})] \\ &= \sum_{S\subseteq N} \frac{(n-s)!(s-1)!}{n!} \left[f\left(\mu_1\left(\bigcup_{j\in S} A_j\right),\cdots,\mu_k\left(\bigcup_{j\in S} A_j\right)\right)\right. \\ &\quad \left. - f\left(\mu_1\left(\bigcup_{j\in S} A_j - A_i\right),\cdots,\mu_k\left(\bigcup_{j\in S} A_j - A_i\right)\right)\right] \\ &= \sum_{i\in S\subseteq N} \frac{(n-s)!(s-1)!}{n!} \sum_{r=1} \mu_r(A_i) \cdot \\ &\quad \cdot f_r\left(\mu_1\left(\bigcup_{j\in S} A_j\right) - \theta\mu_1(A_i),\cdots,\mu_k\left(\bigcup_{j\in S} A_j\right) - \theta\mu_k(A_i)\right). \end{aligned}$$

The last equation follows from the mean value theorem, $0 < \theta < 1$.

Hence

$$w_m(A_i) = \sum_{r=1}^{k} \mu_r(A_i) \sum_{i \in S \subseteq N} \frac{(n-s)!(s-1)!}{n!} f_r(\cdots) \tag{2.1}$$

so that if we could prove that

$$\lim_{m\to\infty} \sum_{i \in S \subseteq N} \frac{(n-s)!(s-1)!}{n!} f_r(\cdots) = \int_0^1 f_r(a_1 x, \cdots, a_k x)\, dx, \tag{2.2}$$

uniformly in i, it would follow that

$$w_m(T) \to \sum_{r=1}^{k} \mu_r(T) \int_0^1 f_r(a_1 x, \cdots, a_k x)\, dx. \tag{2.3}$$

Indeed, because of (2.2), for every $\varepsilon > 0$ there exists an M such that

$$m > M \Rightarrow \left| \sum_{i \in S \subseteq N} \frac{(n-s)!(s-1)!}{n!} f_r(\cdots) - \int_0^1 f_r(a_1 x, \cdots, a_k x)\, dx \right| < \varepsilon;$$

hence

$$\begin{aligned}
&\left| w_m(T) - \sum_{r=1}^{k} \mu_r(T) \int_0^1 f_r(a_1 x, \cdots, a_k)\, dx \right| \\
&= \left| \sum_{j=1}^{t} w_m(A_{i_j}) - \sum_{r=1}^{k} \sum_{j=1}^{t} \mu_r(A_{i_j}) \int_0^1 f_r(a_1 x, \cdots, a_k x)\, dx \right| \\
&= \left| \sum_{r=1}^{k} \sum_{j=1}^{t} \mu_r(A_{i_j}) \sum_{i_j \in S \subseteq N} \frac{(n-s)!(s-1)!}{n!} f_r(\cdots) \right. \\
&\qquad \left. - \sum_{r=1}^{k} \sum_{j=1}^{t} \mu_r(A_{i_j}) \int_0^1 f_r(a_1 x, \cdots, a_k x)\, dx \right| \\
&\leqq \sum_{r=1}^{k} \sum_{j=1}^{t} \mu_r(A_{i_j}) \left| \sum_{i_j \in S \subseteq N} \frac{(n-s)!(s-1)!}{n!} f_r(\cdots) \right. \\
&\qquad \left. - \int_0^1 f_r a_1 x, \cdots, a_k x)\, dx \right| \\
&\leqq \sum_{r=1}^{k} \mu_r(T) \cdot \varepsilon. \quad \left(T = \bigcup_{j=1}^{t} A_{i_j} \right)
\end{aligned}$$

It remains to prove (2.2).

If $i \in S \subseteq N$, then $S' = S - \{i\} \subset N - \{i\} = N'$ and

$$\frac{(n-s)!(s-1)!}{n!} = \frac{1}{n} \cdot \frac{(n-1-(s-1))!(s-1)!}{(n-1)!}.$$

There are

$$\binom{n-1}{s-1}$$

sets in N' having exactly $s-1$ elements. Recalling the definition of the Riemann integral, it follows that it suffices (for proving (2.2)) to show that

$$\text{(2.4)} \quad \lim_{m\to\infty} \left\{ \sum_{i \in S \subseteq N} \frac{(n-s!(s-1)!}{(n-1)!} f_r(\ldots) - f_r\left(a_1\frac{s-1}{n-1}, \cdots, a_k\frac{s-1}{n-1}\right)\right\} = 0,$$

uniformly in i.

But $$\mu_r\left(\bigcup_{j\in S} A_j\right) - \theta\mu_r(A_i) = \sum_{j \in S-\{i\}} \mu_j(A_i) + (1-\theta)\mu_r(A_i).$$

Hence, we may reformulate our problem as follows: Given a continuous function g on the rectangle $0 \leqq x_i \leqq a_i$, $i = 1, \cdots, k$, and given a set of partitions

$$\left\{ a_i = \sum_{j=1}^{n-1} \alpha_j^i, \qquad i = 1, \cdots, k \right\},$$

show that

$$\lim_{\max|\alpha_{ij}|\to 0} \left\{ \sum_{\substack{S\subseteq N' \\ |S|=s-1}} \frac{(n-s)!(s-1)!}{(n-1)!} g\left(\sum_{j\in S}\alpha_j^1, \cdots, \sum_{j \in S}\alpha_j^k\right) - g\left(\frac{s-1}{n-1}a_1, \cdots, \frac{s-1}{n-1}a_k\right)\right\} = 0.$$

We shall use some concepts of probability theory. Our sample space will be the set of all subsets S of N' having $s-1$ members, giving every element S of the sample space the same probability

$$\frac{1}{\binom{n-1}{s-1}} = \frac{(n-s)!(s-1)!}{(n-1)!}.$$

Define a random variable $X^{(i)}(S)$ by $X^{(i)}(S) = \sum_{j\in S}\alpha_j^i$. Then

$$E(X^{(i)}) = \frac{s-1}{n-1}a_i.$$

According to the Chebyshev inequality,

$$\text{(2.5)} \qquad \Pr\left\{\left| X^{(i)} - \frac{s-1}{n-1}a_i \right| \geqq t\right\} \leqq \frac{\operatorname{Var}(X^{(i)})}{t^2}.$$

Using a well-known formula [1, p. 188, Ex. 16 with $n_i = 1$] we find that

$$\mathrm{Var}(X^{(i)}) = \frac{a^2(s-1)(n-s)}{n-2}$$

where

$$a^2 = \frac{\sum_{j=1}^{n-1} (\alpha_j^i)^2}{n-1} - \left[\frac{\sum_{j=1}^{n-1} \alpha_j^i}{n-1} \right]^2$$

$$= \frac{1}{n-1} \sum_{j=1}^{n-1} \alpha_j^i \cdot \left(\alpha_j^i - \frac{\sum_{k=1}^{n-1} \alpha_k^i}{n-1} \right).$$

If $\max |\alpha_j^i| < \varepsilon$, then $\frac{\sum_{k=1}^{n-1} \alpha_k^i}{n-1} < \varepsilon$, and $a^2 \leqq \frac{1}{n-1} \cdot \sum_{j=1}^{n-1} \alpha_j^i \varepsilon = \frac{\varepsilon a_i}{n-1}$.

Therefore,

$$\mathrm{Var}(X^{(i)}) \leqq \frac{\varepsilon}{n-1} a_i \frac{(s-1)(n-s)}{n-2} \leqq 2\varepsilon a_i. \tag{2.6}$$

Hence $\Pr(|X^{(i)} - a_i \frac{s-1}{n-1}| \geqq t) \leqq \frac{2\varepsilon a_i}{t^2}$, and if we set $\varepsilon = t^3$ we find that $\Pr(|X^{(i)} - a_i \frac{s-1}{n-1}| \geqq t) \leqq Ct$; C is a constant which does not depend on s or n. The function g is continuous on a closed rectangle, and therefore, is also bounded and uniformly continuous. Since we can make t arbitrarily small our assertion follows.

References

1. W. Feller, *Probability Theory and its Applications*, Vol. 1, John Wiley, 1950 (First edition).
2. P. R. Halmos, *Measure Theory*, Van Nostrand, 1950.
3. L. S. Shapley, A value for n-person games, *Contributions to the theory of games* 2, Ann. Math. Studies **28**, pp. 307–318.

The Hebrew University of Jerusalem
and
The Israel Institute for Biological Research